UDC 539.3

STRESS DISTRIBUTION IN THE ZONE OF ABRUPT CHANGE OF ELASTIC PROPERTIES OF AN INHOMOGENEOUS MATERIAL

PMM Vol. 43, No. 4, 1979, pp. 760-764

V. P. PLEVAKO
(Khar'kov)

(Received May 10, 1978)

In bodies containing inclusions of different material and, in particular in those of laminated structure, the elastic properties of the medium may change abruptly from one finite value to another from point to point. This is accompanied by an abrupt change of some stress tensor components. The relation between stresses at two adjacent points located on different sides of a discontinuity surface of elastic properties of the material is established in this paper.

1. The three-dimensional problem. Let us consider a body of arbitrary form with an anisotropy of the most general kind. The body is subjected to constant or time-dependent forces. We denote by S one of the surfaces inside the body along which the functions that determine the mechanical properties of the material become discontinuous at its various points. We select on surface S some point M and introduce in the analysis the orthogonal curvilinear coordinates ξ, η, ζ such that surface $\zeta(x, y, z) = \zeta^*$ merges with S at $M(\xi^*, \eta^*, \zeta^*)$ and in some of its neighborhood. We have to determine the stresses at point $M_2(\xi^*, \eta^*, \zeta^* + 0)$, when the stresses at point $M_1(\xi^*, \eta^*, \zeta^* - 0)$ lying next to M_2 on the other side of surface S are known.

If we assume the presence of the elastic potential, the formulas of the generalized Hooke's law in the selected coordinate system is of the form

$$\begin{aligned} \varepsilon_{\xi\xi} &= c_{11}\sigma_{\xi\xi} + c_{12}\sigma_{\eta\eta} + c_{13}\sigma_{\zeta\zeta} + c_{14}\sigma_{\xi\eta} + c_{15}\sigma_{\xi\zeta} + c_{16}\sigma_{\eta\zeta} \\ \varepsilon_{\eta\eta} &= c_{12}\sigma_{\xi\xi} + c_{22}\sigma_{\eta\eta} + \ldots\ldots\ldots\ldots + c_{26}\sigma_{\eta\zeta} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varepsilon_{\eta\zeta} &= c_{16}\sigma_{\xi\xi} + c_{26}\sigma_{\eta\eta} + \ldots\ldots\ldots\ldots + c_{66}\sigma_{\eta\zeta} \end{aligned} \tag{1.1}$$

where $c_{kl} = c_{kl}(\xi, \eta, \zeta)$ are the material elastic parameters that become discontinuous along S, and σ_{ij}, ε_{ij} $(i, j = \xi, \eta, \zeta)$ are stresses and strains.

The considered body may, obviously, be assumed to consist of two bodies bonded along surface S.

Owing to complete bonding the conditions

$$u_i^- = u_i^+, \quad \sigma_{i\zeta}^- = \sigma_{i\zeta}^+ \quad (i = \xi, \eta, \zeta) \tag{1.2}$$

must be satisfied in the contact zone by the displacements u_i and stresses $\sigma_{i\zeta}$. Here and in what follows the superscripts minus and plus denote, respectively, functions at points M_1 and M_2. It is assumed that no concentrated forces act between these points.

Thus the stresses $\sigma_{i\zeta}$ $(i = \xi, \eta, \zeta)$ are equal at points M_1 and M_2. Consequently, we omit henceforth the plus and minus signs at these. Since stresses $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$, $\sigma_{\xi\eta}$

do not appear in formula (1.2), only they can become discontinuous on S and change abruptly between points M_1 and M_2.

Thus, if the stress tensor components are known at point M_1, the problem reduces to the determination of $\sigma_{\xi\xi}^+, \sigma_{\eta\eta}^+, \sigma_{\xi\eta}^+$.

For the determination of strains from (1.2) we have

$$\varepsilon_{\xi\xi}^- = \varepsilon_{\xi\xi}^+, \quad \varepsilon_{\eta\eta}^- = \varepsilon_{\eta\eta}^+, \quad \varepsilon_{\xi\eta}^- = \varepsilon_{\xi\eta}^+ \tag{1.3}$$

Similar relations for $\varepsilon_{\xi\zeta}, \varepsilon_{\eta\zeta}, \varepsilon_{\zeta\zeta}$ may not be necessarily satisfied, since the formulas for their computation contain derivatives of u_i $(i = \xi, \eta, \zeta)$ in a direction normal to S but, obviously, generally

$$\partial u_i^- / \partial \zeta \neq \partial u_i^+ / \partial \zeta$$

Substituting the expressions in (1.1) into (1.3) we obtain for the determination of $\sigma_{\xi\xi}^+, \sigma_{\eta\eta}^+, \sigma_{\xi\eta}^+$ a system of algebraic equations whose solution in matrix form is

$$\sigma = C^{-1} C_1 \sigma_1 \tag{1.4}$$

$$C = \begin{Vmatrix} c_{11}^+ & c_{12}^+ & c_{14}^+ \\ c_{12}^+ & c_{22}^+ & c_{24}^+ \\ c_{14}^+ & c_{24}^+ & c_{44}^+ \end{Vmatrix}, \qquad \begin{matrix} \sigma = \mathrm{col}\,\{\sigma_{\xi\xi}^+, \sigma_{\eta\eta}^+, \sigma_{\xi\eta}^+\} \\ \sigma_1 = \mathrm{col}\,\{\sigma_{\xi\xi}^-, \sigma_{\eta\eta}^-, \sigma_{\xi\eta}^-, \sigma_{\zeta\zeta}, \sigma_{\xi\zeta}, \sigma_{\eta\zeta}\} \end{matrix}$$

$$C_1 = \begin{Vmatrix} c_{11}^- & c_{12}^- & c_{14}^- & c_{13}^- - c_{13}^+ & c_{15}^- - c_{15}^+ & c_{16}^- - c_{16}^+ \\ c_{12}^- & c_{22}^- & c_{24}^- & c_{23}^- - c_{23}^+ & c_{25}^- - c_{25}^+ & c_{26}^- - c_{26}^+ \\ c_{14}^- & c_{24}^- & c_{44}^- & c_{34}^- - c_{34}^+ & c_{45}^- - c_{45}^+ & c_{46}^- - c_{46}^+ \end{Vmatrix}$$

In the particular case of isotropic material relations (1.4) are considerably simplified, and may be written as

$$\sigma_{\xi\xi}^+ = \beta_1 \sigma_{\xi\xi}^- + \beta_2 \sigma_{\eta\eta}^- + \beta_3 \sigma_{\zeta\zeta}, \quad \sigma_{\xi\eta}^+ = \beta_4 \sigma_{\xi\eta}^- \tag{1.5}$$

$$\sigma_{\eta\eta}^+ = \beta_2 \sigma_{\xi\xi}^- + \beta_1 \sigma_{\eta\eta}^- + \beta_3 \sigma_{\zeta\zeta}$$

$$\beta_1 = \frac{G^+(1 - \nu^- \nu^+)}{G^-(1 - \nu^+)(1 + \nu^-)}, \quad \beta_2 = \frac{G^+(\nu^+ - \nu^-)}{G^-(1 - \nu^+)(1 + \nu^-)}$$

$$\beta_3 = \frac{\nu^+}{1 - \nu^+} - \frac{G^+ \nu^- (1 + \nu^+)}{G^-(1 - \nu^+)(1 + \nu^-)}, \quad \beta_4 = \frac{G^+}{G^-}$$

where G is the shear modulus and ν the Poisson ratio.

Formulas (1.5) imply that

$$\sigma_{\xi\xi}^+ - \sigma_{\eta\eta}^+ = \frac{G^+}{G^-}(\sigma_{\xi\xi}^- - \sigma_{\eta\eta}^-) \tag{1.6}$$

If $\nu^+ = \nu^- = \nu$, formulas (1.5) assume the form

$$\sigma_{kk}^+ = \sigma_{kk}^- + \frac{G^+ - G^-}{G^-}\left(\sigma_{kk}^- - \frac{\nu}{1 - \nu}\sigma_{\zeta\zeta}\right) \quad (k = \xi, \eta), \quad \sigma_{\xi\eta}^+ = \frac{G^+}{G^-}\sigma_{\xi\eta}^- \tag{1.7}$$

Validity of the derived formulas can be tested on the example of any problem (in particular on the problem in [1]) on the state of stress of a composite body, for which a rigorous analytic solution is available. These formulas enable us to evaluate each of the stress tensor components, as well as the effect of the "jump" in the material elastic properties on the character of the abrupt change of the state of stress of the body at transition through surface S. They are also of interest in the development of a more precise theory of reinforcement.

The obtained formulas make possible a reduction of the computation process.

Thus for determining the stress fields in laminated systems it is necessary to determine for each layer with ordinal number n expressions of the form [2, 3]

$$\sum_{m=-\infty}^{+\infty} e^{im\beta} \int_0^\infty J_m(\alpha r)\, s^{(n)}(z, \alpha, m)\, d\alpha \tag{1.8}$$

where r, β, z are cylindrical coordinates, $J_m(\alpha r)$ is a Bessel function of the first kind of order m, and $s^{(n)}$ is some function of z and parameters m and α, whose form is determined by the type of loading and the inhomogeneity of the structural material.

Each function $s^{(n)}$ depends on the unknowns $C_k^{(n)} = C_k^{(n)}(\alpha, m)$ $(k = 1, 2, \ldots)$ which are determined by a system of algebraic equations. Owing to the unwieldiness of obtained expressions, it is difficult to discern the existence of formulas (1.4) or (1.5)–(1.7). Because of this, in earlier investigations of state of stress in laminated bodies, the laborious computation based on expressions (1.8) were carried out separately for each of points M_1 and M_2.

Formulas (1.4) or (1.5) and (1.7) make possible a considerable reduction of the computation time, since it is sufficient to determine stresses only at point M_1 using expressions of form (1.8) and, then, using the above formulas, determine the abruptly changing stress tensor components at the adjacent point M_2 lying on the opposite side of the bond surface of layers.

As an example of application of derived formulas, let us consider the problem encountered in the calculation of pavements. An inhomogeneous layer of thickness h rests on a homogeneous isotropic half-space $z > h$ to which it is bonded. The variation with depth of the [pavement] layer shear modulus conforms to the hyperbolic law $G(z) = G_0(1 + cz)^b$, while the shear modulus of the half-space $G^+ = \text{const}$. The Poisson coefficient of the structure $\nu = 1/3$. The layer is subjected to stresses q normal to its surface which are evenly distributed over the area of a circle of radius δ. We have to determine the normal stresses $\sigma_r^\pm$, $\sigma_\beta^\pm$ and σ_z at points M_1 and M_2 lying on the z-axis on opposite sides of the plane $z = h$ (Fig. 1).

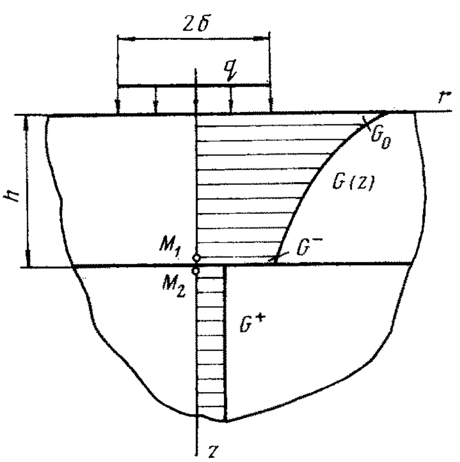

Fig. 1

An analytic solution of this problem appeared in [3]. Formulas presented here were used for determining the stresses $\sigma_r^- = \sigma_\beta^-$ and σ_z at point M_1 of the layer. Some of the obtained results are tabulated below. The left- and right-hand columns of figures relate, respectively, to $G^+ / G^- = 0.2$ and $G^+ / G^- = 0.8$, where G^- is the shear modulus at the base of the layer. Formulas (1.7) with ($\xi = r$, $\eta = \beta$, $\zeta = z$, $\sigma_{\xi\xi} = \sigma_r$, $\sigma_{\eta\eta} = \sigma_\beta$, $\sigma_{\zeta\zeta} = \sigma_z$) were used for calculating stresses $\sigma_r^+ = \sigma_\beta^+$ at point M_2.

2. **The plane problem.** We assume that at every point of a linearly elastic body it is possible to draw perpendicularily to the z-axis a plane such that any two directions symmetric about it are equaivalent as regards the body elastic properties. We further assume that the mechanical properties of the material are functions of coordinates x and y, and that S is one of the cylindrical surfaces on which these properties become discontinuous. We shall consider the case of plane deformation in the xOy-plane, and denote by L the curve of intersection of that plane with S.

G_0/G^-	δ/h	$\sigma_r^-/q\cdot 10^3$		$-\sigma_z/q\cdot 10^3$		$\sigma_r^+/q\cdot 10^3$	
2	0.6	404	46	179	313	9.2	5.5
2	1.0	693	25	363	573	−6.6	−37.3
2	1.4	829	−44	527	743	−45.0	−110
10	0.6	332	48	134	252	12.8	13.2
10	1.0	618	52	286	489	9.2	−7.3
10	1.4	800	9	431	665	−12.4	−59.3

As in the preceding problem, we take some point of L, and introduce in the xOy-plane the curvilinear coordinates ξ, η, with the curve $\eta(x, y) = \eta^*$ merging with L at point $M(\xi^*, \eta^*)$ and in some of its neighborhood.

Thus the problem consists of determining stresses at point $M_2(\xi^*, \eta^* + 0)$ from known stresses at point $M_1(\xi^*, \eta^* - 0)$ which lies next to it on the other side of curve L.

In this case conditions (1.2) assume the form

$$u_i^- = u_i^+, \quad \sigma_{i\eta}^- = \sigma_{i\eta}^+ \quad (i = \xi, \eta) \tag{2.1}$$

which implies that only the component $\sigma_{\xi\xi}$ can abruptly change its sign at transition through L. The remaining components $\sigma_{\eta\eta}$ and $\sigma_{\xi\eta}$ are constant, hence we discard in what follows their superscripts plus and minus. It follows from (2.1) that the relative strains

$$\varepsilon_{\xi\xi}^- = \varepsilon_{\xi\xi}^+ \tag{2.2}$$

Moreover

$$\varepsilon_{\xi\xi} = c_{11}\sigma_{\xi\xi} + c_{12}\sigma_{\eta\eta} + c_{13}\sigma_z + c_{14}\sigma_{\xi\eta} \tag{2.3}$$

$$\varepsilon_z = c_{13}\sigma_{\xi\xi} + c_{23}\sigma_{\eta\eta} + c_{33}\sigma_z + c_{34}\sigma_{\xi\eta} = 0. \tag{2.4}$$

Substituting in (2.4) for σ_z its expression determined from (2.3) we obtain

$$\varepsilon_{\xi\xi} = d_{11}\sigma_{\xi\xi} + d_{12}\sigma_{\eta\eta} + d_{14}\sigma_{\xi\eta}$$
$$d_{11} = c_{11} - \frac{c_{13}^2}{c_{33}}, \quad d_{12} = c_{12} - \frac{c_{13}c_{23}}{c_{33}}, \quad d_{14} = c_{14} - \frac{c_{13}c_{34}}{c_{33}} \tag{2.5}$$

Note that in the generalized state of stress $d_{kl} = c_{kl}$. The substitution of relation (2.5) into formula (2.2) yields for the determination of $\sigma_{\xi\xi}^+$ the equation

$$\sigma_{\xi\xi}^+ = \frac{d_{11}^-}{d_{11}^+}\sigma_{\xi\xi}^- + \frac{d_{12}^- - d_{12}^+}{d_{11}^+}\sigma_{\eta\eta} + \frac{d_{14}^- - d_{14}^+}{d_{11}^+}\sigma_{\xi\eta}$$

which for an isotropic material assumes the form

$$\sigma_{\xi\xi}^{+}=\frac{1-\nu^{-}}{1-\nu^{+}}\frac{G^{+}}{G^{-}}\sigma_{\xi\xi}^{-}+\frac{G^{-}\nu^{+}-G^{+}\nu^{-}}{(1-\nu^{+})G^{-}}\sigma_{\eta\eta}$$

In the case of the generalized state of stress $\nu^{\pm}/(1-\nu^{\pm})$ must be substituted for $\nu^{\pm}$.

The results of the present investigation can be extended to problems involving temperature effects by including in formulas (1.1) terms that take into account such effects.

REFERENCES

1. Plevako, V. P., A concentrated force inside half-spaces bonded together. Osnovaniia, Fundamenty i Mekhanika Gruntov, No. 3, 1969.

2. Plevako, V. P., Inhomogeneous layer bonded to a half-space under the action of internal and external forces. PMM, Vol. 38, No. 5, 1974.

3. Plevako, V. P. and Kerevich, A. A., Inhomogeneous layer bonded to a half-space subjected to external forces. Prikl. Mekh., Vol. 13, No. 5, 1977.

Translated by J.J.D.